Tempered Radon measures

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Let $\mu$ be a Radon measure in $\mathbb{R}^n$. It has the following properties:

$$\mu(K) < +\infty$$

for compact sets $K \subset \mathbb{R}^n$,

$$\mu(O) = \sup \left\{ \mu(K) : K \subset O, \ K \text{ compact} \right\}$$

for open sets $O \subset \mathbb{R}^n$,

$$\mu(A) = \inf \left\{ \mu(O) : A \subset O, \ O \text{ open} \right\}$$

for any $\mu$–measurable $A \subset \mathbb{R}^n$.

$$T_\mu : \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x) \mu(dx), \ \varphi \in D(\mathbb{R}^n).$$

**Definition (Tempered Radon measure).** $\mu$ is said to be tempered if $T_\mu \in S'(\mathbb{R}^n)$.  

1
Theorem.

(i) A Radon measure $\mu$ in $\mathbb{R}^n$ is tempered if, and only if, there is a real number $\beta$ such that $\langle x \rangle^\beta \mu$ is finite.

(ii) Let $\mu$ be a tempered Radon measure in $\mathbb{R}^n$. Let $j \in \mathbb{N}$,

$$A_j = \left\{ x : 2^{j-1} \leq |x| \leq 2^{j+1} \right\}, \quad A_0 = \left\{ x : |x| \leq 2 \right\}.$$

Then for some $c > 0$, $\alpha \geq 0$,

$$\mu(A_k) \leq c2^{k\alpha} \quad \text{for all } k \in \mathbb{N}_0.$$
Finite measures

Definition (Positive distribution). $f \in S'(\mathbb{R}^n)$ is called a positive tempered distribution if

$$f(\varphi) \geq 0 \quad \text{for any} \quad \varphi \in S(\mathbb{R}^n) \text{ with } \varphi \geq 0.$$

The positive cone $B^s_{pq}(\mathbb{R}^n)$ is the collection of all positive $f \in B^s_{pq}(\mathbb{R}^n)$.

Theorem. Let $M(\mathbb{R}^n)$ be the collection of all finite Radon measures. Then

$$M(\mathbb{R}^n) = \overset{+}{B}^0_{1\infty}(\mathbb{R}^n)$$

$$\mu(\mathbb{R}^n) \sim ||\mu|B^0_{1\infty}(\mathbb{R}^n)||, \quad \mu \in M(\mathbb{R}^n).$$
Corollary. Let $f \in L_1(\mathbb{R}^n)$ and $f(x) \geq 0$ a.e. Then
\[
\|f|L_1(\mathbb{R}^n)\| \sim \|f|B_{1\infty}^0(\mathbb{R}^n)\|.
\]

Proposition. There are functions $f_j \in L_1(\mathbb{R}^n)$ with
\[
\text{supp } f_j \subset \{y : |y| \leq 1\}, \quad j \in \mathbb{N},
\]
such that $\{f_j\}$ is a bounded set in $B_{1\infty}^0(\mathbb{R}^n)$, but
\[
\|f_j|L_1(\mathbb{R}^n)\| \to \infty \quad \text{if } j \to \infty.
\]

Corollary. Not any characteristic function of a measurable sub-set of $\mathbb{R}^n$ is a pointwise multiplier in $B_{1\infty}^0(\mathbb{R}^n)$. 