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I. Two-Weight Estimates for Fourier Operators and Bernstein Inequality

Abstract

The norm estimation problem for Fourier operators acting from $L^p_w(\mathbb{T})$ to $L^q_v(\mathbb{T})$ where $1 < p \leq q < \infty$ was investigated. These results were generalized to the two-dimensional case and applied to obtain generalizations of the Bernstein inequality for trigonometric polynomials of one and two variables. Also, the rates of convergence of Cesaro and Abel-Poisson means of functions $f \in L^p_w(\mathbb{T})$ were estimated in the case $p = q$ and $v \equiv w$. The generalized Bernstein inequality applied to estimate the order of best trigonometric approximation of the derivative of functions $f \in L^p_w(\mathbb{T})$ in the space $L^q_v(\mathbb{T})$. 
1 Introduction

Let $\mathbb{T}$ be the interval $[-\pi, \pi]$. A $2\pi$-periodic nonnegative integrable function $w : \mathbb{T} \to \mathbb{R}$ is called a weight function. We denote by $L^p_w(\mathbb{T})$, $1 \leq p < \infty$ the Banach function space of all measurable $2\pi$-periodic functions $f$, for which

$$\|f\|_{p,w} = \left( \int_{\mathbb{T}} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

The uniform boundedness problem of Cesaro and Abel-Poisson means of functions $f \in L^p_w(\mathbb{T})$ ($1 < p < \infty$) was studied in [10] and [7]. In the paper [10], the complete characterization of the weights $w$, for which the Cesaro and Abel-Poisson means are bounded as operators from $L^p_w(\mathbb{T})$ to $L^p_w(\mathbb{T})$ has been done. Later on B. Muckenhoupt showed that the condition referred in [10] is equivalent to the condition $w \in A_p(\mathbb{T})$, that is

$$\sup_I \frac{1}{|I|} \int_I w(x) \, dx \left( \frac{1}{|I|} \int_I w^{1-p'}(x) \, dx \right)^{p-1} < \infty, \quad (1)$$

where $p' = p/(p - 1)$ and the supremum is taken over all intervals whose lengths are not greater than $2\pi$ (see [7]).

In two-weighted setting, B. Muckenhoupt has shown that (see [8]) the necessary and sufficient condition for the uniform boundedness of the Abel-Poisson means as a sequence of operators from $L^p_w(\mathbb{T})$ to $L^p_v(\mathbb{T})$ is

$$\sup_I \frac{1}{|I|} \int_I v(x) \, dx \left( \frac{1}{|I|} \int_I w^{1-p'}(x) \, dx \right)^{p-1} < \infty. \quad (2)$$
The set of all pairs \((v, w)\) of weights with the condition (??) is also denoted by \(A_p(T)\). It was shown in [1] that the condition (??) is also necessary and sufficient for the boundedness of the Cesaro mean from \(L^p_w(T)\) to \(L^p_v(T)\) where \(\alpha > 0\).

Let

\[
 f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{3}
\]

be the Fourier series of the function \(f \in L^1(T)\). Let \(\sigma^\alpha_n(\cdot, f) (\alpha > 0)\) be the Cesaro mean of the series (??), that is

\[
 \sigma^\alpha_n(x, f) = \frac{1}{\pi} \int_T f(x + t) K^\alpha_n(t) \, dt,
\]

where

\[
 K^\alpha_n(t) = \sum_{k=0}^{n} \frac{A^{\alpha-1}_{n-k} D_k(t)}{A^\alpha_n}
\]

is the Fejer kernel and

\[
 D_k(t) = \frac{\sin \left( k + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}
\]

is the Dirichlet kernel, with

\[
 A^\alpha_n = \binom{n + \alpha}{\alpha} \approx \frac{n^\alpha}{\Gamma(\alpha + 1)}.
\]

Let also \(U_r(\cdot, f) (0 \leq r < 1)\) be the Abel-Poisson mean of the function \(f\), that is

\[
 U_r(x, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x - t) f(t) \, dt,
\]
where
\[
P_r(t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}
\]
is the Poisson kernel.

In the present paper, we investigated the problem estimation of norms of the operators \( \sigma_n^\alpha (\cdot, f) (\alpha > 0) \) and \( U_r (\cdot, f) (0 \leq r < 1) \) from \( L^p_w (\mathbb{T}) \) to \( L^q_v (\mathbb{T}) \) where \( 1 < p \leq q < \infty \). These results were generalized to the two-dimensional case and applied to obtain generalizations of the Bernstein inequality for trigonometric polynomials of one and two variables. Also, we estimated the rates of convergence of the Cesaro and Abel-Poisson means of functions \( f \in L^p_w (\mathbb{T}) \) in the norm of \( L^q_v (\mathbb{T}) \), and we estimated the order of best trigonometric approximation of the derivative of functions \( f \in L^p_w (\mathbb{T}) \) in the space \( L^q_v (\mathbb{T}) \).

2 One-dimensional case

Let us introduce the certain class of pairs of weight functions.

**Definition. 2.1.** Let \( 1 < p \leq q < \infty \). A pair of weight functions \((v, w)\) is said to be of class \( A_{p,q} (\mathbb{T}) \) if the \( A_{p,q} \)-condition

\[\sup_I \left( \frac{1}{|I|} \int_I v(x) \, dx \right)^{1/q} \left( \frac{1}{|I|} \int_I w^{1-p'}(x) \, dx \right)^{1/p'} < \infty \quad \text{(4)}\]

holds, where the supremum is taken over all intervals with lengths not more than \( 2\pi \).

The following statements are true.

**Theorem 2.2.** Let \( 1 < p \leq q < \infty \). The inequality

\[\| \sigma_n^\alpha (\cdot, f) \|_{q,v} \leq c \, n^{\frac{1}{p} - \frac{1}{q}} \| f \|_{p,w}, \quad \alpha > 0 \quad \text{(5)}\]
holds for arbitrary \( f \in L^p_w(T) \), where the constant \( c \) does not depend on \( n \) and \( f \), if and only if \((v,w) \in A_{p,q}(T)\).

**Theorem 2.3.** Let \( 1 < p \leq q < \infty \). The necessary and sufficient condition for the validity of the inequality

\[
\|U_r (\cdot, f)\|_{q,v} \leq c \left(1 - r\right)^{\frac{1}{q} - \frac{1}{p}} \|f\|_{p,w}
\]

for arbitrary \( f \in L^p_w(T) \), where the constant \( c \) does not depend on \( r \) and \( f \), is \((v,w) \in A_{p,q}(T)\).

**Theorem 2.4.** Let \( 1 < p < \infty \). Then

\[
\lim_{n \to \infty} \|\sigma_n^\alpha (\cdot, f) - f\|_{p,v} = 0, \quad \alpha > 0,
\]

for every \( f \in L^p_w(T) \) if and only if \( w \in A_{p,q}(T) \).

The analogous statement in true for Abel-Poisson means.

Let \( w \in A_p(T) \) and \( f \in L^p_w(T) \). It is well known that for the Steklov mean

\[
f_h(x) = \frac{1}{h} \int_{x-h}^{x+h} f(t) \, dt, \quad h > 0
\]

the inequality

\[
\|f_h\|_{p,w} \leq c \, \|f\|_{p,w}
\]

holds, where the constant \( c \) is independent of \( h \) and \( f \). Starting from this we define the modulus of continuity for the function \( f \in L^p_w(T) \) as

\[
\Omega (\delta, f)_{p,w} = \sup_{h \leq \delta} \|f - f_h\|_{p,w}, \quad \delta \geq 0.
\]

**Theorem 2.5.** Let \( 1 < p < \infty \), \( w \in A_p(T) \) and \( f \in L^p_w(T) \). Then there exist a constant \( c \), which does not depend on \( n \) and \( f \), such that the estimate

\[
\|\sigma_n (\cdot, f) - f\|_{p,w} \leq c \, n \, \Omega \left(\frac{1}{n}, f\right)_{p,w}
\]

(7)
holds.

**Theorem 2.6.** Let \( 1 < p < \infty, \ w \in A_p (\mathbb{T}) \) and \( f \in L_{w}^{p} (\mathbb{T}) \). Then there exist a constant \( c \), which does not depend on \( r \) and \( f \), such that the estimate

\[
\|U_r (\cdot, f) - f\|_{p,w} \leq c \frac{1}{1 - r} \Omega (1 - r, f)_{p,w}
\]

holds.

3 Two-dimensional case

Let \( \mathbb{T}^2 = \mathbb{T} \times \mathbb{T} \) and \( w \) be weight function on \( \mathbb{T}^2 \). We denote by \( L_{w}^{p} (\mathbb{T}^2) \), \( 1 \leq p < \infty \), the space of functions \( f (x, y) \) which are \( 2\pi \)-periodic with respect to each variable, such that

\[
\|f\|_{p,w} = \left( \int_{\mathbb{T}^2} |f(x, y)|^p w(x, y) \, dx \, dy \right)^{1/p} < \infty.
\]

Let the function \( f \in L^{1} (\mathbb{T}^2) \) has the Fourier series

\[
f (x, y) \sim \sum_{m,n=0}^{\infty} \lambda_{mn} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny) + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny
\]

where

\[
\lambda_{mn} = \begin{cases} 
\frac{1}{4}, & m = n = 0, \\
\frac{1}{2}, & m = 0, n > 0 \text{ or } m > 0, n = 0, \\
1, & m > 0, n > 0.
\end{cases}
\]
Let also
\[
\sigma_{mn}^{(\alpha,\beta)}(x, y, f) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} S_{ij}(x, y, f)}{A_m A_n}, \quad (\alpha, \beta > 0)
\]
be the Cesaro means of the function \(f\), where \(S_{ij}(\cdot, \cdot, f)\) are the partial sums of the series \((??)\), and
\[
U_{r\rho}(x, y, f) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, s) P_r(x - t) P_\rho(y - s) \, dt \, ds, \quad 0 \leq r, \rho < 1
\]
be the Abel-Poisson means of \(f\).

**Definition 3.1.** The pair \((v, w)\) is said to belong to the class \(A_{p,q}(\mathbb{T}^2, \mathbb{J})\) if the condition
\[
\sup_{J \in \mathbb{J}} \left( \frac{1}{|J|} \int_J v(x, y) \, dx \, dy \right)^{1/q} \left( \frac{1}{|J|} \int_J w^{1-p'}(x, y) \, dx \, dy \right)^{1/p'} < \infty
\]
holds, where \(\mathbb{J}\) denote the set of all rectangles with the sides parallel to the coordinate axes.

In this section we will give the two-dimensional analogues of Theorem 2.2 and Theorem 2.3.

**Theorem 3.2.** Let \(1 < p \leq q < \infty\). The condition \((v, w) \in A_{p,q}(\mathbb{T}^2, \mathbb{J})\) is necessary and sufficient for validity of the inequality
\[
\left\| \sigma_{mn}^{(\alpha,\beta)}(\cdot, \cdot, f) \right\|_{q,v} \leq c (mn)^{\frac{1}{p'} - \frac{1}{q}} \|f\|_{p,w}, \quad \alpha > 0, \beta > 0 \quad (10)
\]
for every \(f \in L^p_w(\mathbb{T}^2)\), where the constant \(c\) is independent of \(m, n\) and \(f\).
In the case $p = q$ Theorem 3.2 was proved in [?].

**Theorem 3.3.** Let $1 < p \leq q < \infty$. The inequality

$$\| U_{r \rho} (\cdot, \cdot, f) \|_{q,v} \leq c (1 - r)^{\frac{1}{q} - \frac{1}{p}} (1 - \rho)^{\frac{1}{q} - \frac{1}{p}} \| f \|_{p,w}$$

(11)

holds for arbitrary $f \in L^p_w (\mathbb{T}^2)$, where the constant $c$ does not depend on $r$, $\rho$ and $f$, if and only if $(v, w) \in A_{p,q} (\mathbb{T}^2, J)$. To prove these theorems we need the boundedness of the generalized multiple Steklov means

$$f_{hk}^{\gamma} (x, y) = \frac{1}{(hk)^{1-\gamma}} \int_{x-hy-k}^{x+hy+k} \int |f (t, s)| \, dt \, ds, \quad 0 \leq \gamma < 1, \quad h, k > 0.$$

**Theorem 3.4.** ([?]) Let $1 < p \leq q < \infty$, $(v, w) \in A_{p,q} (\mathbb{T}^2, J)$ and $\gamma = 1/p - 1/q$. Then the inequality

$$\| f_{hk}^{\gamma} \|_{q,v} \leq c \| f \|_{p,w}$$

(12)

holds for every $f \in L^p_w (\mathbb{T}^2)$.

4 Generalizations of the Bernstein’s inequality

The Bernstein inequality which is very important in approximation theory, states that for any trigonometric polynomial $T_n (x)$ of degree $\leq n$,

$$\| T_n' \|_p \leq c \, n \| T_n \|_p, \quad 1 \leq p \leq \infty.$$

By virtue of the inequality (??) we can prove the generalization of the Bernstein’s inequality:

**Theorem 4.1.** Let $1 < p \leq q < \infty$, $(v, w) \in A_{p,q} (\mathbb{T})$ and
Let \( T_n(x) \) be a trigonometric polynomial of degree at most \( n \). Then the inequality

\[
\| T'_n \|_{q,v} \leq c \ n^{1+\frac{1}{p}-\frac{1}{q}} \| T_n \|_{p,w}
\]

(13)

holds, where the constant \( c \) is not depend on \( n \) and \( T_n \). Also for the conjugate trigonometric polynomial \( \tilde{T}_n \), we have

\[
\left\| \left( \tilde{T}_n \right)' \right\|_{q,v} \leq c \ n^{1+\frac{1}{p}-\frac{1}{q}} \| T_n \|_{p,w}.
\]

(14)

Also we have the extended Bernstein inequality for trigonometric polynomials of two variables.

**Theorem 4.2.** Let \( 1 < p \leq q < \infty \) and \((v, w) \in A_{p,q} \left( T^2, J \right)\). Then for every trigonometric polynomial \( T_{mn}(x, y) \) of degree \( \leq m \) with respect to \( x \) and of degree \( \leq n \) with respect to \( y \), the inequality

\[
\left\| \frac{\partial^2 T_{mn}}{\partial x \partial y} \right\|_{q,v} \leq c \ (mn)^{1+\frac{1}{p}-\frac{1}{q}} \| T_{mn} \|_{p,w}
\]

(15)

holds, where the constant \( c \) is not depend on \( m, n \) and \( T_{mn} \).

**References**


II. ON THE MAXIMAL AND FOURIER OPERATORS IN WEIGHTED LEBESGUE SPACES WITH VARIABLE EXPONENT

Sawyer type two-weighted criteria

Let $J$ be a subinterval of $\mathbb{R}$. Suppose that $p$ is measurable function on $J$ with the condition

$$1 < p_-(J) \leq p(x) \leq p_+(J) < \infty,$$

where

$$p_-(J) := \inf_J p; \quad p_+(J) := \sup_J p.$$

Suppose also that $\rho$ is an almost everywhere positive locally integrable function on $J$, i.e. $\rho$ is a weight. We say that a measurable function $f : J \to \mathbb{R}$, belongs to $L^{p(x)}_{\rho}(J)$ (or $L^{p_-(J)}_{\rho}(J)$) if

$$S_{p,\rho}(f) = \int_J |f(x)\rho(x)|^{p(x)} dx < \infty.$$

It is known that $L^{p(x)}_{\rho}(J)$ is a Banach space with the norm

$$\|f\|_{L^{p(x)}_{\rho}(J)} = \inf \{ \lambda > 0 : S_{p,\rho}(f/\lambda) \leq 1 \}.$$

If $p = \text{const}$, then $L^{p(x)}_{\rho}(J)$ coincides with the classical Lebesgue space with the weight $\rho$. Further, if $\rho \equiv 1$, then we use the symbol $L^{p(x)}(J)$ for $L^{p(x)}_{\rho}(J)$.

For some basic properties of $L^{p(x)}$ spaces we refer, e.g., to [4-6].

We say that $p : J \to \mathbb{R}$ satisfies the Dini-Lipschitz (log-Hölder continuity) condition on $J$ ( $p \in DL(J)$) if there
exists a positive constant $A$ such that

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x - y|}; \quad x, y \in J; \quad |x - y| \leq 1/2.$$ 

A weight function $\rho$ satisfies the doubling condition on $J$ ($\rho \in DC(J)$) if there exists a positive constant $b$ such that

$$\int_{I(x, 2r)} \rho \leq b \int_{I(x, r)} \rho$$

for all $x \in J$ and $r > 0$, where $I(x, r) := (x - r, x + r)$.

The following statements are true:

**Theorem 1.** Let $1 < p_-(T) \leq p(x) \leq p_+(T) < \infty$ and let $p \in DL(T)$. If $(w(\cdot))^{-p'(')}$ satisfies the doubling condition on $T$, then the following conditions are equivalent:

i) The Hardy-Littlewood maximal operator is bounded from $L^{p(')}(T)$ to $L^{p(')}(T)$

ii) there exists a constant $c > 0$ such that

$$\int_I (v(x))^{p(x)} \left( M(w^{-p'(')}(\cdot)\chi_I(\cdot))(x) \right)^{p(x)} dx \leq c \int_I w^{-p'(x)} dx$$

for an arbitrary interval $I \subset T$.

iii) $\| \sup_n \sigma_n^\alpha(f, \cdot) \|_{L^{p(')}_v(T)} \leq c \| f \|_{L^{p(')}_w(T)}$

iv) $\| \sup_{0<r<1} u_r(f, \cdot) \|_{L^{p(')}_v(T)} \leq c \| f \|_{L^{p(')}_w(T)}$

**Theorem 2.** Let $1 < p_-(T) \leq p(x) \leq p_+(T) < \infty$ and let $p \in DL(T)$. Suppose that $(w(\cdot))^{-p'(')} \in DC(T)$ and the condition (1) holds with $v = w$. Then for arbitrary $f \in L^{p(')}_w(T)$ we have

$$\lim_{n \to \infty} \| \sigma_n^\alpha(f, \cdot) - f(\cdot) \|_{L^{p(')}_w(T)} = 0$$
and
\[
\lim_{r \to 1} \|u_r(f, \cdot) - f(\cdot)\|_{L^p_w(T)} = 0.
\]

Two-weight estimates for the Cesàro means enable us to obtain the extended Bernstein inequality for the derivative of trigonometric polynomial and its conjugate in two-weighted setting.

**Theorem 3.** Let \(1 < p_-(T) \leq p(x) \leq p_+(T) < \infty\) and let \(p \in DL(T)\). Suppose that \((w(\cdot))^{-p'(\cdot)} \in DC(T)\) and condition (1) is satisfied. Then for an arbitrary trigonometric polynomial \(t_n(x)\) and its conjugate \(\tilde{t}_n(x)\) we have
\[
\|t'_n v\|_{L^p(\cdot)(T)} \leq cn \|t_n w\|_{L^p(\cdot)(T)}
\]
and
\[
\|\tilde{t}'_n v\|_{L^p(\cdot)(T)} \leq cn \|t_n w\|_{L^p(\cdot)(T)}.
\]

For special pairs \((v, w)\) the above mentioned results were obtained in [1]. For the constant \(p\) we refer to [2].

Now we discuss the Hardy-Littlewood maximal operators on the line.

Let \(M_{R^+}\) and \(M_R\) be maximal operators given by
\[
(M_{R^+}f)(x) = \sup_{r>0} \frac{1}{2r} \int_{(x-r,x+r) \cap R^+} |f(t)| dt, \quad x \in R^+,
\]
\[
(M_Rf)(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| dt, \quad x \in R
\]
respectively.

**Definition.** An exponent \(p(x)\) is said to be of class \(DL(R)_{\infty} (DL(R^+)_{\infty}\), if there exists the \(\lim_{x \to \infty} p(x) = p(\infty) \left(\lim_{x \to +\infty} p(x) = p(\infty)\right)\)
and a positive constant \(c > 0\) such that
\[
|p(x) - p(\infty)| \leq \frac{c}{\ln(e + |x|)}
\]
for arbitrary \( x \in R \) \( (x \in R_+) \).

We have the following statements:

**Theorem 4.** Let \( 1 < p-(R_+) \leq p(x) \leq p+(R_+) < \infty \) and let \( p \in DL(R_+) \cap DL(R_+). \) Suppose that \((w(\cdot))^{-p'(\cdot)} \in DC\). Then \( M_{R+} \) is bounded from \( L^p_w(R_+) \) to \( L^p_v(R_+) \) if and only if there is a positive constant \( c \) such that for all bounded subintervals \( I \) of \( R_+ \),

\[
\|M_{R+}(w^{-p'(\cdot)} \chi_I)\|_{L^p_v(I)} \leq c \|w^{1-p'(\cdot)}(\cdot)\|_{L^p_v(I)} < \infty.
\]

**Theorem 5.** Let \( 1 < p-(R) \leq p(x) \leq p+(R) < \infty \) and let \( p \in DL(R) \cap DL(R) \). Suppose that \((w(\cdot))^{-p'(\cdot)} \in DC\). Then for the boundedness of \( M_R \) from \( L^p_w(R) \) to \( L^p_v(R) \) it is necessary and sufficient that there exists a positive constant \( c \) such that for all bounded subintervals \( I \) of \( R \),

\[
\|M_R(w^{-p'(\cdot)} \chi_I)\|_{L^p_v(I)} \leq c \|w^{1-p'(\cdot)}(\cdot)\|_{L^p_v(I)} < \infty.
\]

Finally we notice that two-weight Sawyer-type criteria for maximal functions in Lebesgue spaces defined on finite intervals were announced in [3].

In the sequel by \( V \) we denote the class of all measurable functions \( f : R^1 \rightarrow R^1 \) for which

\[
\int_{-\infty}^{\infty} \frac{f(x)}{(1 + |x|)^2} dx < \infty.
\]

**Theorem 6.** Let the conditions of Theorem 5 hold with \( v = w \). Then for arbitrary \( f \in L^p_w(R^1) \cap V \) we have

\[
\lim_{t \to 0} \|f - U_t(\cdot, f)\|_{L^p_w} = 0
\]

where

\[
U_t(x, f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)t}{t^2 + (x - y)^2} dy.
\]
REFERENCES


III. INTEGRAL TRANSFORMS IN MORREY SPACES ON QUASIMETRIC MEASURE SPACES

Constant exponent case

1. Introduction

The main purpose of this part of the talk is to give the boundedness results for fractional integral operators in (weighted) Morrey spaces defined on quasimetric measure spaces. We derive Sobolev, trace and two-weight inequalities for fractional integrals. In particular, we generalize: a) D. Adams trace inequality; b) the theorem by E. M. Stein and G. Weiss regarding the two-weight inequality for the Riesz potentials; c) Sobolev-type inequality. We emphasize that in the most cases the derived conditions are necessary and sufficient for appropriate inequalities.

In [KMo] a complete description of non-doubling measure $\mu$ guaranteeing the boundedness of fractional integral operator $I_{\alpha}$ (see the next section for the definition) from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1 < p < q < \infty$, was given. We notice that this result for potentials on Euclidean spaces was derived in [Ko]. In [KMo], theorems of Sobolev and Adams type for fractional integrals defined on quasimetric measure spaces were established. For the boundedness of fractional integrals on metric measure spaces we refer also to [GCG]. Some two-weight norm inequalities for fractional operators on $\mathbb{R}^n$ with non-doubling measure were studied in [GCM]. Further, in the paper [KM1] necessary and sufficient conditions on measure $\mu$ governing the inequality of Stein-Weiss type on nonhomogeneous spaces were established.
The boundedness of the Riesz potential in Morrey spaces defined on Euclidean spaces was studied in [Pe] and [Ad1]. The same problem for fractional integrals on $\mathbb{R}^n$ with non-doubling measure was investigated in [Sa, Ta].

Finally we mention that necessary and sufficient conditions for the boundedness of maximal operators and Riesz potentials in the local Morrey-type spaces were derived in [BG], [BGG].

Several results on boundedness of integral transforms in Morrey spaces in Euclidean space are obtained in [Gu] and [Ta] papers.

It should be emphasized that the results of this work are new even for Euclidean spaces.

Constants (often different constants in the same series of inequalities) will generally be denoted by $c$ or $C$.

2. Preliminaries

We assume that $X := (X, \rho, \mu)$ be a topological space, endowed with a complete measure $\mu$ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (quasimetric) $\rho : X \times X \rightarrow [0, \infty)$ satisfying the conditions:

1. $\rho(x, y) > 0$ for all $x \neq y$, and $\rho(x, x) = 0$ for all $x \in X$;
2. there exists a constant $a_0 \geq 1$, such that $\rho(x, y) \leq a_0 \rho(y, x)$ for all $x, y \in X$;
3. there exists a constant $a_1 \geq 1$, such that $\rho(x, y) \leq a_1 (\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$.

We assume that the balls $B(a, r) := \{x \in X : \rho(a, x) < r\}$ are measurable, for $a \in X$, $r > 0$, and $0 \leq \mu(B(a, r)) < \infty$. 
For every neighborhood $V$ of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty$, $\mu\{a\} = 0$, and $B(a, r_2) \setminus B(a, r_1) \neq \emptyset$, for all $a \in X$, $0 < r_1 < r_2 < \infty$.

The triple $(X, \rho, \mu)$ will be called quasimetric measure space.

Let $0 < \alpha < 1$. We consider the fractional integral operators $I_\alpha$, and $K_\alpha$ given by

$$I_\alpha f(x) := \int_X f(y) \rho(x, y)^{\alpha-1} d\mu(y),$$

$$K_\alpha f(x) := \int_X f(y)(\mu B(x, \rho(x, y)))^{\alpha-1} d\mu(y),$$

for suitable $f$ on $X$.

Suppose that $\nu$ is another measure on $X$, $\lambda \geq 0$ and $1 \leq p < \infty$. We deal with the Morrey space $L^{p,\lambda}(X, \nu, \mu)$, which is the set of all functions $f \in L^p_{\text{loc}}(X, \nu)$ such that

$$\|f\|_{L^{p,\lambda}(X, \nu, \mu)} := \sup_B \left( \frac{1}{\mu(B)^{\lambda}} \int_B |f(y)|^p d\nu(y) \right)^{1/p} < \infty,$$

where the supremum is taken over all balls $B$.

If $\nu = \mu$, then we have the classical Morrey space $L^{p,\lambda}(X, \mu)$ with measure $\mu$. When $\nu = \mu$ and $\lambda = 0$, then $L^{p,\lambda}(X, \nu, \mu) = L^p(X, \mu)$ is the Lebesgue space with measure $\mu$.

Further, suppose that $\beta \in \mathbb{R}$. We are also interested in weighted Morrey space $M^{p,\lambda}_{\beta}(X, \mu)$ which is the set of all $\mu$-measurable functions $f$ such that

$$\|f\|_{M^{p,\lambda}_{\beta}(X, \mu)} := \sup_{a \in X; r > 0} \left( \frac{1}{r^\lambda} \int_{B(a, r)} |f(y)|^p \rho(a, y)^{\beta} d\mu(y) \right)^{1/p} < \infty.$$

If $\beta = 0$, then we denote $M^{p,\lambda}_{\beta}(X, \mu) := M^{p,\lambda}(X, \mu)$. 
We say that a measure $\mu$ satisfies the growth condition ($\mu \in (GC)$), if there exists $C_0 > 0$ such that $\mu(B(a, r)) \leq C_0 r$; further, $\mu$ satisfies the doubling condition ($\mu \in (DC)$) if $\mu(B(a, 2r)) \leq C_1 \mu(B(a, r))$ for some $C_1 > 1$. If $\mu \in (DC')$, then $(X, \rho, \mu)$ is called a space of homogeneous type (SHT). A quasimetric measure space $(X, \rho, \mu)$, where the doubling condition might be failed, is also called a non-homogeneous space.

The measure $\mu$ on $X$ satisfies the reverse doubling condition ($\mu \in (RDC)$) if there are constants $\eta_1$ and $\eta_2$ with $\eta_1 > 1$ and $\eta_2 > 1$ such that
\[
\mu(B(x, \eta_1 r)) \geq \eta_2 \mu(B(x, r)). \tag{1}
\]

It is known (see e.g. [St,To], p. 11) that if $\mu \in (DC')$, then $\mu \in (RDC')$.

The next statements is from [KMo] and [Ko] in the case of Euclidean spaces.

**Theorem A.** Let $(X, \rho, \mu)$ be a quasimetric measure space. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1$. Then $I_\alpha$ is bounded from $L^p(X)$ to $L^q(X)$ if and only if there exists a positive constant $C$ such that
\[
\mu(B(a, r)) \leq Cr^{s}, \quad s = \frac{pq(1 - \alpha)}{pq + p - q}, \tag{2}
\]
for all $a \in X$ and $r > 0$.

**Corollary B.** Let $(X, \rho, \mu)$ be a quasimetric measure space, $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Then $I_\alpha$ is bounded from $L^p(X)$ to $L^q(X)$ if and only if $\mu \in (GC)$.

The latter statement by different proof was also derived in [GCG] for metric spaces.
Theorem C. Let $(X, \rho, \mu)$ be an SHT. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Assume that $\nu$ is another measure on $X$. Then $K_\alpha$ is bounded from $L^p(X, \mu)$ to $L^q(X, \nu)$ if and only if

$$\nu B \leq c(\mu B)^{q(1/p - \alpha)}$$

for all balls $B$ in $X$.

3. Main results (constant exponent case)

In this section we formulate the main results in constant exponent case. We begin with the case of an SHT.

**Theorem 3.1.** Let $(X, \rho, \mu)$ be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \gamma_1 < 1 - \alpha p$ and $\gamma_2/q = \gamma_1/p$. Then $K_\alpha$ is bounded from $L^{p,\gamma_1}(X, \mu)$ to $L^{q,\gamma_2}(X, \nu, \mu)$ if and only if there is a positive constant $c$ such that

$$\nu(B) \leq c \mu(B)^{q(1/p - \alpha)},$$

(3)

for all balls $B$.

The next statement is a consequence of Theorem 3.1.

**Theorem 3.2.** Let $(X, \rho, \mu)$ be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \gamma_1 < 1 - \alpha p$ and $\gamma_2/q = \gamma_1/p$. Then for the boundedness of $K_\alpha$ from $L^{p,\gamma_1}(X, \mu)$ to $L^{q,\gamma_2}(X, \nu, \mu)$ it is necessary and sufficient that

$$q = p/(1 - \alpha p).$$

For non-homogeneous spaces we have the following statements:

**Theorem 3.3.** Let $(X, \rho, \mu)$ be a non-homogeneous space with the growth condition. Suppose that $1 < p \leq q < \infty$, $1/p - 1/q \leq \alpha < 1$ and $\alpha \neq 1/p$. Suppose
also that \( p\alpha - 1 < \beta < p - 1 \), \( 0 < \lambda_1 < \beta - \alpha p + 1 \) and \( \lambda_1 q = \lambda_2 p \). Then \( I_\alpha \) is bounded from \( M_{\beta_1}^{p, \lambda_1}(X, \mu) \) to \( M_{\gamma_2}^{q, \lambda_2}(X, \mu) \), where \( \gamma = q(1/p + \beta/p - \alpha) - 1 \).

**Theorem 3.4.** Suppose that \((X, \rho, \mu)\) is a quasimetric measure space and \( \mu \) satisfies condition (2). Let \( 1 < p < q < \infty \). Assume that \( 0 < \alpha < 1 \), \( 0 < \lambda_1 < p/q \) and \( s\lambda_1/p = \lambda_2/q \). Then the operator \( I_\alpha \) is bounded from \( M_{p, \lambda_1}^{p, \lambda_1}(X, \mu) \) to \( M_{q, \lambda_2}^{q, \lambda_2}(X, \mu) \).

**Variable Exponent Case**

4. **MAXIMAL FUNCTIONS AND POTENTIALS IN VARIABLE EXPONENT MORREY SPACES ON QUASIMETRIC SPACES WITH NON-DOUBLING MEASURE**

Let \((X, \rho, \mu)\) be a quasimetric measure spaces and let \( \mu X < +\infty \). For a measurable function \( p : X \to \mathbb{R}^1 \) we introduce the notations:

\[
p_-(E) = \text{ess inf}_{x \in E} p(x) \quad \text{and} \quad p_+(E) = \text{ess sup}_{x \in E} p(x)
\]

**Definition.** A measurable function \( p : X \to \mathbb{R}^1 \) with the condition \( 1 < p_- \leq p(x) \leq p_+ < \infty \) is said to be of class \( \mathcal{P}(N), \ N \geq 1 \), if there exists a positive constant \( c \) such that

\[
\mu(NB)^{p_-(B) - p_+(B)} \leq c
\]

for arbitrary ball \( B \subset X \).
The variable exponent Morrey space is defined as a set of all measurable functions $f$, for which

$$
\| f \|_{M^{p(\cdot)}_{q(\cdot)}(X)_{N}} = \sup_{x \in X} \left( \mu B(x, Nr) \right)^{\frac{1}{p(x)} - \frac{1}{q(x)}} \| f \|_{L^{q(\cdot)}B(x,r)}
$$

We study the boundedness problems for the following the operators:

$$(\mathcal{M}^{N} f)(x) = \sup_{B \ni x} \frac{1}{\mu(NB)} \int_{B} |f(x)| d\mu, \quad N = a_{1}(1 + 2a_{0})$$

- the maximal function and

$$(I_{\alpha(x)} f)(x) = \int_{X} \frac{f(y)}{\rho(x, y)^{1-\alpha(x)}} d\mu(y), \quad x \in X, \quad 0 < \alpha_{-} \leq \alpha_{+} < 1,$$

- the fractional integral with variable parameter $\alpha(\cdot)$.

**Theorem 4.1.** Let $p$ and $q$ belong to $\mathcal{P}(N)$. Suppose that $1 < q_{-} \leq q(x) \leq p(x) \leq p_{+} < \infty$. Then the maximal operator $\mathcal{M}^{N}$ is bounded from $M^{p(\cdot)}_{q(\cdot)}(X)_{N}$ to $M^{p(\cdot)}_{q(\cdot)}(X)_{Na}$ where $a = a_{1}(a_{0} + 1) + 1$.

**Theorem 4.2 (Sobolev type).** Let $N := a_{1}(1 + 2a_{0})$, $1 < q_{-} \leq q(x) \leq p(x) \leq p_{+} < \infty$, $1 < t_{-} \leq t(x) \leq s(x) \leq s_{+} < \infty$. Suppose that $0 < \alpha_{-} \leq \alpha_{+} < \frac{1}{p_{-}}$, $s(x) = \frac{p(x)}{1-\alpha(x)p(x)}$, $\frac{t(x)}{s(x)} = \frac{q(x)}{p(x)}$ and that $p, q, \alpha \in \mathcal{P}(N)$. Suppose also that the measure $\mu$ satisfies the growth condition: there exists a positive constant $b$ such that for all $x \in X$ and $r > 0$,

$$
\mu(B(x, r)) \leq br.
$$

Then the operator $I_{\alpha(x)}$ is bounded from $M^{p(\cdot)}_{q(\cdot)}(X)_{N}$ to $M^{s(\cdot)}_{t(\cdot)}(X)_{Na}$.

The proofs of these Theorems are based on the following boundedness Theorem of maximal function $\mathcal{M}^{N}$ in $L^{p(\cdot)}(X, \rho, \mu)$. 
**Theorem 4.3.** Suppose that $\mu(X) < \infty$, $1 < p_- \leq p(x) \leq p(x) \leq p_+ < \infty$. Let the following condition hold there exists a positive constant $c$ such that for all $x$ and $r$, $x \in X$, $0 < r < \text{diam}(X)$,

$$\tag{1} (\mu B(x, N r))^{p_-(B(x,r)) - p(x)} \leq c$$

Then there exists a constant $b > 0$ such that

$$\|\mathcal{M}^N f\|_{L^p(\cdot)(X)} \leq b\|f\|_{L^p(\cdot)(X)}$$

for arbitrary $f \in L^p(\cdot)(X)$.