Equivalent norms on grand and small Lebesgue spaces

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Outline

Grand Lebesgue space

Small Lebesgue space

Interpolation theorem

Compactness of Sobolev imbedding

Equivalent norms on grand and small Lebesgue spaces
Definition (of a class of weights)

Let us denote $W$ the class of all functions $\varphi : (0, +\infty) \to (0, 1]$ which are increasing and there are constants $1 < C_1 \leq C_2 < \infty$ such that $C_1 \varphi(2^{-k}) \leq \varphi(2^{-k+1}) \leq C_2 \varphi(2^{-k})$ for all $k \in \mathbb{N}$. Furthermore, we assume that $\varphi(t) = 1$ for $t > 1$. 
Definition
Let the measure of $\Omega$ is equal to 1 and $1 < p < +\infty$. Then for $\varphi \in \mathcal{W}$ we define the grand Lebesgue space

$$L^{p)}_{\varphi}(\Omega) = \{ f \in \mathcal{M}_0(\Omega) : \|f\|_{p,\varphi} = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{p-\varepsilon} < +\infty \}.$$
Definition
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$$L^p_\varphi(\Omega) = \{ f \in \mathcal{M}_0(\Omega) : \|f\|_{p,\varphi} = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{p-\varepsilon} < +\infty \}.$$ 

This space with the weight $\varphi(\varepsilon) = \varepsilon^{1/(p-\varepsilon)}$ was introduced in the paper by T. Iwaniec and C. Sbordone in 1992 where they study the integrability of the Jacobian under minimal hypotheses.
Remark

Let $\Omega$ has finite measure, $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$L^p(\Omega) \subset L^p_{\varphi}(\Omega) \subset L^{p-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, p - 1).$$
Remark

Let $\Omega$ has finite measure, $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$L^p(\Omega) \subset L^p(\Omega) \subset L^{p-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, p - 1).$$

Definition

If we have a function $f$ on $\Omega$ and $k \in \mathbb{N}$ we define $f_k^*$ to be the restriction of $f^*$ to the interval $[2^{-2^k+1}, 2^{-2^k-1+1}]$. 
Theorem

Let $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$\|f\|_{p),\varphi} \approx \sup_{k \in \mathbb{N}} \varphi(2^{-k}) \|f_{k}\|_{p}$$

and

$$\|f\|_{p),\varphi} \approx \sup_{0 < t < 1} \varphi\left(\frac{1}{1 - \log t}\right) \left(\int_{t}^{1} (f^*(s))^p ds\right)^{1/p}.$$
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Definition
Let the measure of $\Omega$ is equal to 1 and $1 < p < +\infty$. Then for $\varphi \in \mathcal{W}$ we define the small Lebesgue space $L^{(p)}(\Omega)$ as the space of all measurable functions $f$ on $\Omega$ such that the following norm is finite:

$$\|f\|_{(p,\varphi)} = \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varphi(\varepsilon) \|f_k\|_{p+\varepsilon}.$$
Definition

Let the measure of $\Omega$ is equal to 1 and $1 < p < +\infty$. Then for $\varphi \in \mathcal{W}$ we define the small Lebesgue space $L^{(p)}_\varphi(\Omega)$ as the space of all measurable functions $f$ on $\Omega$ such that the following norm is finite:

$$\|f\|_{(p,\varphi)} = \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \frac{1}{\varphi(\varepsilon)} \|f_k\|_{p+\varepsilon}.$$

This space with the weight $\varphi(\varepsilon) = \varepsilon^{1/(p'-\varepsilon)}$ was found by A. Fiorenza in 2000 as the associate space of the grand Lebesgue space $L^{p'}_\varphi(\Omega)$.
Remark

Let $\Omega$ has finite measure, $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$L^{p+\varepsilon}(\Omega) \subset L^{p}(\varphi(\Omega)) \subset L^{p}(\Omega), \quad \forall \varepsilon > 0.$$
Theorem

Let $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$\|f\|_{(p, \varphi)} \approx \sum_{k \in \mathbb{N}} \frac{\|f_k^*\|_p}{\varphi(2^{-k})}$$

and, moreover, $\|f\|_{(p, \varphi)}$ is equivalent to the following norm:

$$\int_0^1 \left( (1 - \log t) \varphi \left( \frac{1}{1 - \log t} \right) \right)^{-1} \left( \int_0^t (f^*(s))^p \, ds \right)^{1/p} \frac{dt}{t}.$$
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Definition
Let $\mathcal{X} = (X_0, X_1)$ be a compatible couple, $1 \leq q \leq +\infty$ and let $w$ be a non-negative measurable function on $(0, +\infty)$. Then the space $\mathcal{X}_{w,q}$ consists of all functions $f$ in $X_0 + X_1$ such that the following norm is finite:

$$
\|f\|_{\mathcal{X}_{w,q}} = \left( \int_0^\infty (w(t)K(f, t; \mathcal{X}))^q \frac{dt}{t} \right)^{1/q}, \quad 1 \leq q < \infty,
$$

$$
\|f\|_{\mathcal{X}_{w,\infty}} = \sup_{0 < t < \infty} w(t)K(f, t; \mathcal{X}), \quad q = \infty,
$$

where the K-functional is defined by

$$
K(f, t; \mathcal{X}) = \inf \{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1 \}.
$$
Theorem

Let $1 \leq q < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$L^p_\varphi = (L^q, L^p)_{w, \infty}$$

where $w(t) = \frac{1}{t} \varphi\left(\frac{1}{1 - \log t}\right)$

with equivalence of norms.
Theorem

Let $1 < p < +\infty$ and $\varphi \in \mathcal{W}$. Then

$$L_\varphi^p = (L^p, L^\infty)_{w,1} \quad \text{where} \quad w(t) = \frac{1}{(1 - \log t) \varphi\left(\frac{1}{1 - \log t}\right)}$$

with equivalence of norms.
Theorem

Let \( 1 \leq q_1 < p_1 < +\infty \), \( 1 \leq q_2 < p_2 < +\infty \) and let \( T \) be a bounded linear operator such that

\[
T : L^{q_1}(\Omega_1) \to L^{q_2}(\Omega_2)
\]

and

\[
T : L^{p_1}(\Omega_1) \to L^{p_2}(\Omega_2).
\]

If \( \varphi \in \mathcal{W} \), then \( T : L^{p_1}(\Omega_1) \to L^{p_2}(\Omega_2) \).
Theorem

Let $1 < p_1, p_2 < +\infty$ and let $T$ be a bounded linear operator such that

$$T : L^{p_1}(\Omega_1) \rightarrow L^{p_2}(\Omega_2)$$

and

$$L^\infty(\Omega_1) \rightarrow L^\infty(\Omega_2).$$

If $\varphi \in \mathcal{W}$, then $T : L^{(p_1)}(\Omega_1) \rightarrow L^{(p_2)}(\Omega_2)$. 
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Theorem (A. Fiorenza, J. M. Rakotoson, 2005)

Let us have a connected bounded open domain $\Omega$ in $\mathbb{R}^n$ with the Lipschitz boundary, let $1 < p < n$ and $\varphi \in \mathcal{V}$. Then the imbedding of $W^{1,1}_\varphi(\Omega)$ into $L^{p^*}(\Omega)$ is compact, where $p^* = np/(n - p)$ and

$$W^{1,1}_\varphi(\Omega) = \{ f \in W^{1,1}(\Omega) : \| f \|_1 + \| \nabla f \|_{(p, \varphi)} < +\infty \}.$$
Theorem

Let us have a connected bounded open domain $\Omega$ in $\mathbb{R}^n$ with the Lipschitz boundary and let $1 < p < n$. If $\varphi \in \mathcal{W}$ then the imbedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is compact.
Proof.

Let us have a bounded sequence \( \{f_k\} \) in \( W^{1,p}(\Omega) \).
Proof.

- Let us have a bounded sequence \( \{f_k\} \) in \( W^{1,p}(\Omega) \).
- Fix an increasing sequence \( q_i \nearrow p^* \). Then by the Rellich-Kondrachov theorem we can choose a subsequence \( \{f_k\} \) such that it is cauchy in every \( L^{q_i} \).
Proof.

- Let us have a bounded sequence \( \{ f_k \} \) in \( W^{1,p}(\Omega) \).
- Fix an increasing sequence \( q_i \rightarrow p^* \). Then by the Rellich-Kondrachov theorem we can choose a subsequence \( \{ f_k \} \) such that it is cauchy in every \( L^{q_i} \).
- \( W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \Rightarrow \exists K > 0 : \| f_k \|_{p^*} \leq K, \ k \in \mathbb{N} \).
Proof.

- Let us have a bounded sequence \( \{f_k\} \) in \( W^{1,p}(\Omega) \).
- Fix an increasing sequence \( q_i \uparrow p^* \). Then by the Rellich-Kondrachov theorem we can choose a subsequence \( \{f_k\} \) such that it is cauchy in every \( L^{q_i} \).
- \( W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \Rightarrow \exists K > 0 : \|f_k\|_{p^*} \leq K, \ k \in \mathbb{N} \).
- Choose an arbitrary \( \delta > 0 \). Since \( \lim_{t \to 0^+} \varphi(t) = 0 \) there exists \( \varepsilon_0 > 0 \) such that \( \varphi(\varepsilon_0)K < \delta/4 \).
Proof.

- Let us have a bounded sequence \( \{f_k\} \) in \( W^{1,p}(\Omega) \).
- Fix an increasing sequence \( q_i \nearrow p^* \). Then by the Rellich-Kondrachov theorem we can choose a subsequence \( \{f_k\} \) such that it is cauchy in every \( L^{q_i} \).
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- Take \( l \in \mathbb{N} \) such that \( (i, j \geq l) \Rightarrow \|f_i - f_j\|_{p^*-\varepsilon_0} < \delta/2 \).
Proof.

- Let us have a bounded sequence \( \{f_k\} \) in \( W^{1,p}(\Omega) \).
- Fix an increasing sequence \( q_i \nearrow p^* \). Then by the Rellich-Kondrachov theorem we can choose a subsequence \( \{f_k\} \) such that it is cauchy in every \( L^{q_i} \).
- \( W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \Rightarrow \exists K > 0 : \|f_k\|_{p^*} \leq K, \ k \in \mathbb{N} \).
- Choose an arbitrary \( \delta > 0 \). Since \( \lim_{t \to 0^+} \varphi(t) = 0 \) there exists \( \varepsilon_0 > 0 \) such that \( \varphi(\varepsilon_0)K < \delta/4 \).
- Take \( I \in \mathbb{N} \) such that \( (i, j \geq I) \Rightarrow \|f_i - f_j\|_{p^* - \varepsilon_0} < \delta/2 \).
- For \( i, j \geq I \) we have:
  \[
  \|f_i - f_j\|_{L^{p^*}} = \sup_{0 < \varepsilon < p-1} \varphi(\varepsilon)\|f_i - f_j\|_{p^* - \varepsilon} \\
  \leq \sup_{0 < \varepsilon \leq \varepsilon_0} \ldots + \sup_{\varepsilon_0 < \varepsilon < p-1} \ldots \leq \varphi(\varepsilon_0)2K + \delta/2 \leq \delta.
  \]
Thank you for your attention.